Microscopic Theory of the Photon Recoil of an Atom in a Dielectric

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Abstract

An atom recoils when it undergoes spontaneous decay. In this paper we present a microscopic calculation of the recoil of a source atom imbedded in a dielectric medium. We find that the source atom recoils with the canonical photon momentum $n\hbar k_{0,}$ where n is the index of refraction and $\hbar k_0$ is the photon momentum calculated at the source atom atomic frequency ω_0. We also show explicitly how the energy is conserved with the photon inside the medium.

I. INTRODUCTION

The momentum of a photon in a dispersive medium has been considered by many authors, due to its conceptual and practical importance. One of the issues is how the momentum is conserved when a photon with momentum $\hbar\mathbf{k}$ in the vacuum is scattered from an atom in the medium to a new momentum state $\hbar\mathbf{k}'$ in the vacuum. Should the momentum imparted on the atom be the difference of the momenta in the vacuum $\hbar\mathbf{k}' - \hbar\mathbf{k}$ or the difference of the canonical photon momenta in the medium $n\left(\hbar\mathbf{k}' - \hbar\mathbf{k}\right)$ [6]? One might argue[2] that, assuming the medium is dilute, the atom is localized in the vacuum space between particles of the medium, therefore the photon, before and after it hits the atom, travels in the vacuum and momentum should be conserved in terms of the vacuum momenta. On the other hand [1], one can also argue that it is the macroscopic field of the incident wave that induces and interacts with polarizations and polarization currents of atoms in the medium and therefore the imparted momentum should be the difference of the canonical photon momenta in the medium. Experimentally, this issue has been studied in two systems. One measures the recoil of a mirror immersed in a liquid when the light is reflected from it [3], and more recently [2], a measurement of the recoil frequency of the Bose condensed ^{87}Rb using a two-pulse Ramsey interferometer. Both experiments confirm that atoms recoil according to canonical photon momenta.

Most theoretical studies related to this issue deal mainly with classical fields. Some of the recent work by Loudon [4] and Nelson [5] clarified some issues related to momentum in a dielectric from a quantum and microscopic perspective. Milonni and Boyd [6] consider a case where a source atom imbedded in the medium recoils due to its spontaneous decay. They find that the source atom recoils according to $n\hbar\omega_0/c$, where ω_0 is the atomic frequency and c is the speed of light in the vacuum. Their calculation is based on a Heisenberg Picture approach, where the operator expectation value, $\langle P^2 \rangle$ is calculated to be $n^2\hbar^2\omega_0^2/c^2$. Here we present a similar calculation in the Schrödinger Picture. The calculation in the Schrödinger Picture is particularly revealing because it includes explicitly processes that are responsible for the modification of the momentum imparted by the photon. As we show in the following, the photon travelling in the medium experiences a series of scatterings from medium atoms. Different scattering amplitudes interfere to shift the central frequency of photons in the field. Since the source atom is coupled directly to the field, by momentum conservation, the

source atom recoils according to the modified central frequency of photons. Our calculation is based on a quantum field quantized in the free space, which allows a separate description of the field and the medium, i.e. any wavelength and frequency of the field are calculated unambiguously in the vacuum.

II. MODEL AND THE RECOIL CALCULATION

The calculation is based on a model that we have used previously [7][8]. The source atom, with finite mass M, centered at position $\langle \mathbf{R} \rangle = 0$, has two internal levels, whose frequency separation is denoted by ω_0. The uniformly distributed dielectric atoms have $J = 0$ ground states and $J = 1$ excited states. The frequency separation of the ground and excited states is denoted by ω. We assume the mass of the medium atoms to be infinite, which allows us to ignore recoil of the medium atoms. At $t = 0$, the source atom is excited to the $m = 0$ excited state sublevel with center of mass momentum $\langle P \rangle = 0$, and $\langle \Delta P^2 \rangle \ll (\hbar k_0)^2$, the dielectric atoms are all in their ground states, and there are no photons in the field. The process we consider one in which radiation emitted by the source atom is scattered by dielectric atoms. The medium is modeled to be infinite, i.e. photons are always inside the medium. The vaccum field amplitudes, the medium atoms' excited state amplitudes, and the source atom center of mass motion is caculated as $t \to \infty$. It is assumed that the medium atoms are far detuned from the source atom, $\omega \gg \omega_0$, and also $\omega_0 \gg \gamma$, where γ is the spontaneous decay rate of the source atom.

The free part of Hamiltonian describing such a system is

$$H_0 = \frac{\hbar\omega_0}{2}\sigma_z + \sum_j \sum_{m=-1}^{1} \frac{\hbar\omega}{2}\sigma_z^{(j)}(m) + \hbar\omega_{\mathbf{k}} a_{\mathbf{k}\lambda}^{\dagger} a_{\mathbf{k}\lambda} + \frac{\mathbf{P}^2}{2M} \tag{1}$$

where $\sigma_z = (|2\rangle\langle 2| - |1\rangle\langle 1|)$, $|2\rangle$ and $|1\rangle$ are the $m = 0$ excited and $J = 0$ ground state kets of the source atom, respectively, $\sigma_z^{(j)}(m) = \left(|m\rangle^{(j)}\langle m| - |g\rangle^{(j)}\langle g|\right)$ is the population difference operator between excited state $|J = 1, m\rangle$ and ground state $|J = 0, g\rangle$ of dielectric atom j, and $a_{\mathbf{k}\lambda}$ is the annihilation operator for a photon having momentum $\mathbf{k}$ and polarization λ. We have also included a term describing the external motion of the source atom, where $\mathbf{P}$ is the momentum operator for the source atom. A summation convention is used, in which any repeated symbol on the right hand side of an equation is summed over, unless it also appears on the left-hand side of the equations. The interaction part that couples the field

with the atoms is

$$V = \hbar g_{\mathbf{k}}(\mu_0 \cdot \epsilon_{\mathbf{k}}^{(0)} \sigma_+ a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}_0} - \mu_0 \cdot \epsilon_{\mathbf{k}}^{(0)} a_{\mathbf{k}}^{\dagger} \sigma_- e^{-i\mathbf{k}\cdot\mathbf{R}_0}) + \hbar g'_{\mathbf{k}\lambda} \left[\begin{array}{c} \mu_m \cdot \epsilon_{\mathbf{k}}^{(\lambda)} \sigma_+^{(j)}(m) \left(a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}} - a_{\mathbf{k}\lambda}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{R}} \right) \\ + \mu_m^* \cdot \epsilon_{\mathbf{k}}^{(\lambda)} \sigma_-^{(j)}(m) e^{-i\mathbf{k}\cdot\mathbf{R}} \left(a_{\mathbf{k}\lambda}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{R}} - a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}} \right) \end{array} \right] \quad (2)$$

$$g_{\mathbf{k}} = -i\sqrt{\frac{\omega_{\mathbf{k}}}{2\hbar\epsilon_0 V}} \quad (3)$$

$$g'_{\mathbf{k}\lambda} = -i\sqrt{\frac{\omega_{\mathbf{k}}}{2\hbar\epsilon_0 V}}, \quad (4)$$

where $\sigma_\pm$ are raising and lowering operators for the source atom and $\sigma_\pm^{(j)}(m)$ are raising and lowering operators between the excited state $|J=1,m\rangle$ and the ground state $|J=0,g\rangle$ of dielectric atom j, and μ_0 is the matrix element of the dipole operator for the source atom and μ_m is the matrix element of the medium atom between levels $|J=1,m\rangle$ and $|J=0,g\rangle$. We have taken the reduce matrix element, μ, of dipole moments of the source and medium atoms to be equal to simplify our calculation. The polarization vectors are defined as

$$\epsilon_{\mathbf{k}}^{(1)} = \cos\theta_{\mathbf{k}} \cos\phi_{\mathbf{k}} \hat{\mathbf{x}} + \cos\theta_{\mathbf{k}} \sin\phi_{\mathbf{k}} \hat{\mathbf{y}} - \sin\theta_{\mathbf{k}} \hat{\mathbf{z}} \quad (5)$$

$$\epsilon_{\mathbf{k}}^{(2)} = -\sin\phi_{\mathbf{k}} \hat{\mathbf{x}} + \cos\phi_{\mathbf{k}} \hat{\mathbf{y}}. \quad (6)$$

The source atom interacts only with the z component of the vacuum field. Since the medium atoms are far detuned from the source atom, we include anti-rotating terms in the field-medium atoms' interaction. We have not included such terms for the interaction Hamiltonian between the source atom and the field because we have chosen the initial state to be the source atom excited with no photon in the field.

Instead of writing amplitude equations for different states and then identifying terms corresponding to contributions from different processes [8], we adopt a resolvent approach [9], which allows us to write amplitudes directly from diagrammatic representations of the scattering processes. We want to calculate the recoil energy of the source atom, which includes the contribution from three amplitudes: the source recoiled atom with one photon in the field, the recoiled source atom with one medium atom excited, and the recoiled source

atom with both the field and a medium atom excited. These amplitudes are represented by,

$$b_k = \langle 1, \mathbf{q}; g; \mathbf{k}| \, U(\infty) \, |2, 0; g; 0\rangle \tag{7}$$

$$b_{mj} = \langle 1, \mathbf{q}; m_j; 0| \, U(\infty) \, |2, 0; g; 0\rangle \tag{8}$$

$$b_{mjkk'} = \langle 1, \mathbf{q}; m_j; \mathbf{k}, \mathbf{k}'| \, U(\infty) \, |2, 0; g; 0\rangle \, , \tag{9}$$

where the state of the whole system is labelled with source atom internal states, source atom wave vector, medium atoms internal states, and quantum field wave vectors. For example, $|1, \mathbf{q}; g; \mathbf{k}\rangle$ is a state of the source atom in the ground state $|1\rangle$, with a momentum $\hbar\mathbf{q}$, the medium atoms in the ground state $|g\rangle$, and a photon with a wave vector $\mathbf{k}$ present. We use $|m_j\rangle$ to label the state with a medium atom at position R_j being excited to the $|J = 1, m\rangle$ state. The photon polarizations are not written explicitly in the formulas. It should be understood, in the following perturbative calculations, any intermedia states' photon polarzations are summed over in amplitudes, while the final states' photon polarizations are summed over in probabilities.

The evolution operator can be expressed in terms of the retarded propagator as

$$U(\tau) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE \exp(-iE\tau/\hbar) \, G(E) \tag{10}$$

and the resolvent operator, $G(E + i0)$, is defined as

$$G(E) = \frac{1}{E + i0 - \hat{H}}$$

where H is the total Hamiltonian, E is the energy of the system, and the infinite small imaginary $i0$ prescription is used to yield retarded propagation. In order to carry out a perturbative calculation in orders of the interaction $\hat{V}$, we recast the resolvant operator in the following exact form

$$G(E) = G^{(0)}(E) + G^{(0)}(E) \, \hat{V} G(E) \tag{11}$$

Here $G^{(0)}(E)$ is the zeroth order propagator

$$G^{(0)}(E) = \frac{1}{E + i0 - \hat{H}_0}.$$

The processes that we want to take into account are shown in Fig(1). Let's focus on the calculation of b_k first,

$$b_k = -\frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{-\infty}^{\infty} dE \exp(-iE\tau/\hbar) \, \langle 1, \mathbf{q}; g_m; \mathbf{k}| \, G(E) \, |2, 0; g_m; 0\rangle \, , \tag{12}$$

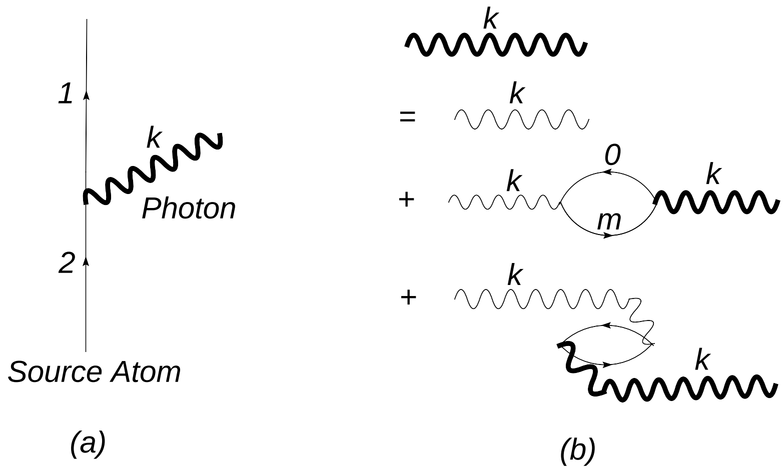

FIG. 1: (a) shows that the excited source atom $|2\rangle$ spontaneously decays by radiating a normalized photon with wave vector $\mathbf{k}$ and goes to the ground state $|1\rangle$ with external momentum $-\hbar\mathbf{k}$. The thick wavy line corresponds to the normalized photon propagator and striaght solid lines correspond to the atom propagator. (b) shows the photon of wavevector $\mathbf{k}$ being scattered by media atoms. We include both the rotating and anti-rotating contributions in the intermedium states. Medium atoms can get excited to $J = 1$ sublevel m by absorbing a photon k and then go to the ground states by radiating photons. They can also get excited by emitting photons and then deexcited by absorbing other photons. In this diagram, the time propagation is from the left to the right. Averaging over the positions of the medium atoms results in conservation of the photon momenta before and after the scattering from medium atoms.

which requires a calculation of the matrix element of the operator. According to equation (11), the resolvant operator can be expanded to first order in $\hat{V}$ as,

$$\begin{aligned}
&\langle 1, \mathbf{q}; g_m; \mathbf{k}| \, G^{(1)} \, |2, 0; g_m; 0\rangle \\
&= \langle 1, \mathbf{q}; g_m; \mathbf{k}| \, G^{(0)}\left(E\right) |1, \mathbf{q}; g_m; \mathbf{k}\rangle \, \langle 1, \mathbf{q}; g_m; \mathbf{k}| \, \hat{V} \, |2, 0; g_m; 0\rangle \, \langle 2, 0; g_m; 0| \, G^{(0)}\left(E\right) |2, 0; g_m; 0\rangle \\
&= \hbar g_k^* \delta_{\mathbf{q},-\mathbf{k}} \left(\mu_m^* \cdot \epsilon_k\right) \frac{1}{E - \hbar\omega_0 + i\hbar n\gamma} \frac{1}{E + i0 - \hbar\omega_q^r - \hbar\omega_k} \mathbf{W}_{m,0}.
\end{aligned} \tag{13}$$

We neglect the possibility that the photon can be scattered by medium atoms in this order. In the calculation, the source atom recoil momentum equals the inverse of the photon momentum, $\mathbf{q} = -\mathbf{k}$ (conservation of momentum), resulting from evaluating the matrix element $\langle \mathbf{p} = \hbar\mathbf{q}| \, e^{i\mathbf{k}\cdot\mathbf{R}} \, |\mathbf{p} = 0\rangle$. We have rearranged the order in the last line of the above

formula (13) so that the first part,

$$\hbar g_k^* \delta_{\mathbf{q},-\mathbf{k}} \left(\mu_m^* \cdot \epsilon_k^{(\lambda)}\right) \frac{1}{E - \hbar\omega_0 + i\hbar n\gamma}, \tag{14}$$

describes the decay of the source atom in the medium. The imaginary part in the dominator $i\hbar n\gamma$ is added to represent spontaneous decay [9]. The modification of γ in the vacume to $n\gamma$ in the medium is related to the process that photons scattered by the media atoms are reabsorbed by the source atom, as shown in papers [7][8]. The second part of Equation(13),

$$\frac{1}{E + i0 - \hbar\omega_q^r - \hbar\omega_k} \mathbf{W}_{m,0}, \tag{15}$$

describes the propagation of the photon, where $\omega_q^r = \hbar q^2/2M$ is the recoil frequency associated with the emission of a phone with wavevector q and $\mathbf{W}_{m,0}$ is the transverse polarization tensor of the photon defined as

$$\begin{aligned} \mathbf{W}_{m,m'} &= \left(\mu_m \cdot \epsilon_k^{(\lambda)}\right)\left(\mu_{m'} \cdot \epsilon_k^{(\lambda)}\right) \\ &= \left(\epsilon_k^{(\lambda)} \otimes \epsilon_k^{(\lambda)}\right)_{m,m'} = \left(1 - \hat{k} \otimes \hat{k}\right)_{m,m'} \end{aligned} \tag{16}$$

where two relationships

$$\mu_m^* \mu_m = \mathbf{1}$$

and

$$\mu_0^* \cdot \epsilon_k = \left(\epsilon_k \cdot \mu_m^*\right) \mathbf{W}_{m,0} \tag{17}$$

have been used.

In order to carry the calculation of the matrix element of the resolvant operator, $\langle 1, \mathbf{q}; g_m; \mathbf{k}| \, G \, |2, 0; g_m; 0\rangle$, to higher order, it is neccessary to consider processes that the photon is scattered by medium atoms. Including these processes modifies the photon propagator(15). For a dilute medium with $N\lambda_0^3 \ll 1$, where N is the density of the medium and $\lambda_0 = 2\pi c/\omega_0$ is the photon wavelength, it is appropriate to make use of the independent scattering approximation. Namely, we need to include only contributions from processes shown in Fig[1b], i.e. ladder diagrams, which amount to a self energy insertion [9] to the

photon propagator(15) in the dominator as the following

$$\sum = \sum_{j,m,k'} \langle 1,\mathbf{q};g;\mathbf{k}| V$$
$$\times \begin{pmatrix} |1,\mathbf{q};m_j;\mathbf{k},\mathbf{k}'\rangle \langle 1,\mathbf{q};m_j;\mathbf{k},\mathbf{k}'| G^{(0)}(E) |1,\mathbf{q};m_j;\mathbf{k},\mathbf{k}'\rangle \langle 1,\mathbf{q};m_j;\mathbf{k},\mathbf{k}'| \\ + |1,\mathbf{q};m_j;0\rangle \langle 1,\mathbf{q};m_j;0| G^{(0)}(E) |1,\mathbf{q};m_j;0\rangle \langle 1,\mathbf{q};m_j;0| \end{pmatrix}$$
$$\times V |1,\mathbf{q};g;\mathbf{k}'\rangle \tag{18}$$

Using the prescription $\sum_{R_j} \rightarrow N\int d\mathbf{R}$ and $\sum_{k'} \rightarrow \frac{V}{(2\pi)^3}\int d\mathbf{k}'$ we can change sums to integrals. The integration over R_j gives $\delta(\mathbf{k}-\mathbf{k}')$ as a result of the translational invariance of the medium. The integration over $\mathbf{k}'$ picks up contributions only at $\mathbf{k}$ and the self energy can be written as

$$\sum = \hbar^2 NV |g_k|^2 \mu^2 \left[\frac{1}{E-\hbar\omega_k^r-\hbar\omega} + \frac{1}{E-\hbar\omega_k^r-\hbar\omega-2\hbar\omega_k}\right] \tag{19}$$

Spontaneous decays of the medium atoms are ignored because medium atoms are far detuned from the source atom. The normalized photon propagator, with the above self energy modification, is

$$\frac{1}{E+i0-\hbar\omega_k^r-\hbar\omega_k-\sum}\mathbf{W} = \frac{1}{E-\hbar\omega_k^r-\left[1-\frac{1}{2}N\alpha(E)\right]\hbar\omega_k}\mathbf{W}, \tag{20}$$

where the polarizability is defined as

$$\alpha(E) = -\frac{4\pi\mu^2}{\Delta(E)} \tag{21}$$

and the detunning

$$\frac{1}{\Delta(E)} \equiv \frac{1}{E-\hbar\omega_k^r-\hbar\omega} + \frac{1}{E-\hbar\omega_k^r-\hbar\omega-2\hbar\omega_k} \tag{22}$$

The self energy insertion(19) brings a correction of order $N\alpha$ to the denominator of the photon propagator, which cannot be obtained from any finite order calculation. Substituting this normalized photon propagator(20) back to the first order formula(13) and using the fact that $\mathbf{W}^2 = \mathbf{W}$, one fines the relevent reolvant matrix element to be

$$\langle 1,-\hbar\mathbf{k};g_m;\mathbf{k};0| G(E+i0) |2,0;g_m;0\rangle$$
$$= \hbar g_k^* (\mu_0^* \cdot \epsilon_k) \frac{1}{E-\hbar\omega_0+in\hbar\gamma} \frac{1}{E-\hbar\omega_k^r-\left(1-\frac{1}{2}N\alpha\right)\hbar\omega_k} \tag{23}$$

To find the transition amplitude in Equation(7), we need to integrate over E according to equation (10). The integration includes contributions from two poles, one at $E = \hbar\omega_k^r + \left(1 - \frac{1}{2}N\alpha\right)\hbar\omega_k$, and one at $E = \hbar\omega_0 - i\hbar n\gamma$. In the limit $\gamma\tau \gg 1$, only the first pole contributes, since the second one has a finite imaginary part and its contribution decays away as $e^{-n\gamma\tau}$. The amplitude of finding a photon with wave number k and source atom with momentum $-\hbar\mathbf{k}$ is then

$$b_k = \lim_{\tau\to\infty} g_k^* \left(\mu_0^* \cdot \epsilon_k\right) \frac{\exp\left[-i\left(\hbar\omega_k^r + \left(1 - \frac{1}{2}N\alpha\right)\omega_k\right)\tau\right]}{\omega_k^r + \left(1 - \frac{1}{2}N\alpha\right)\omega_k - \omega_0 + in\gamma} \tag{24}$$

and the corresponding probability is [10]

$$\left|b_k\right|^2 = \left|g_k\right|^2 \left|\mu_0 \cdot \epsilon_k\right|^2 \frac{1}{\left[\omega_k/n - \left(\omega_0 - n^2\omega_0^r\right)\right]^2 + n^2\gamma^2} \tag{25}$$

with $\omega_0^r = \hbar\omega_0^2/2Mc^2$. We have identified $n = 1 + \frac{1}{2}N\alpha$ in the limit of small $N\alpha$ [11]. The photon frequency centers around $n\left(\omega_0 - n^2\hbar\omega_0^r\right)$. The total probability of finding a photon in the field is just the sum over all the $k's$

$$\sum_k \left|b_k\right|^2 = \frac{4\pi}{3}\frac{2\mu^2}{\left(2\pi\right)^2\hbar c^3}\int d\omega_k \frac{\omega_k^3}{\left[\omega_k/n - \left(\omega_0 - n^2\omega_0^r\right)\right]^2 + n^2\gamma^2} = 1 \tag{26}$$

Note that in working out the integration, we have used the Wigner-Weisskopf approximation. We extend the integration of ω_k to start from minus infinity[14], and take into account only the contribution from the pole at $n\left(\omega_0 - n^2\omega_0^r + in\gamma\right)$[16]. Here the decay rate γ should be evaluated at the photon frequency and it is given by $\gamma = 2\mu^2\omega_k^3/3\hbar c^3$.

Other relevent amplitudes, indicated in Fig[2], can be calculated by the same technique and now consider the processes shown in Fig[2]. Making use of the normalized photon propagator(20), we find the amplitude for a medium atom located at R_j, being excited to a sublevel m, without any photon present to be

$$b_{m_j} = \lim_{\tau\to\infty}\sum_k \left|g_k\right|^2 \mu^2 e^{i\mathbf{k}\cdot\mathbf{R}_j} \frac{\exp\left[-i\left(\hbar\omega_k^r + \left(1 - \frac{1}{2}N\alpha\right)\omega_k\right)\tau\right]}{\omega_k^r + \left(1 - \frac{1}{2}N\alpha\right)\omega_k - \omega_0 + in\gamma} \tag{27}$$

$$\times \frac{1}{\left(1 - \frac{1}{2}N\alpha\right)\omega_k - \omega}\mathbf{W}_{m,0} \tag{28}$$

It proves more useful to work in momentum space, since one can take advantage of the translational invariance of the problem. We label the corresponding momentum state amplitude

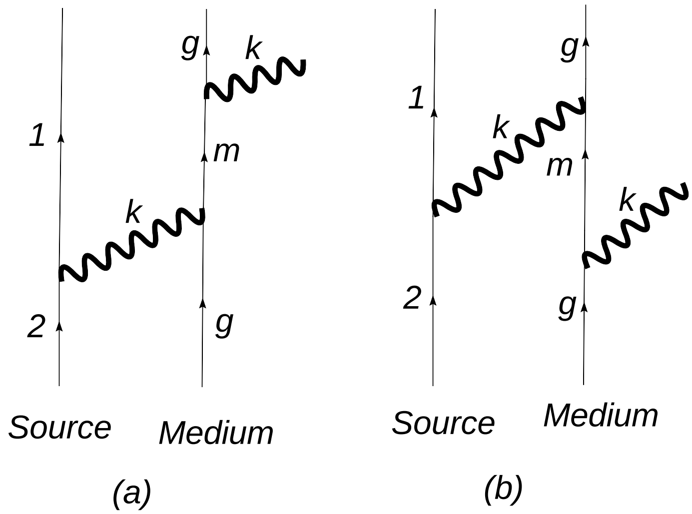

FIG. 2: Medium atoms are excited by emitting or absorbing normalized photons. Figure (a) corresponds to a rotating situation while figure (b) corresponds to an anti-rotating situation.

as $b_m(k)$

$$b_m(k) = \lim_{\tau\to\infty} \sqrt{NV}\mu^2 |g_k|^2 \frac{\exp\left[-i\left(\hbar\omega_k^r + \left(1-\frac{1}{2}N\alpha\right)\omega_k\right)\tau\right]}{\omega_k^r + \left(1-\frac{1}{2}N\alpha\right)\omega_k - \omega_0 + in\gamma} \times \frac{1}{\left(1-\frac{1}{2}N\alpha\right)\omega_k - \omega}\mathbf{W}_{m,0} \tag{29}$$

The probability of finding medium atoms excited is

$$\sum_{m,k} |b_m(k)|^2 = NV\mu^4 \sum_k \frac{|g_k|^4}{\left[\omega_k/n - (\omega_0 - n^2\omega_0^r)\right]^2 + n^2\gamma^2} \times \left[\frac{1}{\left(1-\frac{1}{2}N\alpha\right)\omega_k - \omega}\right]^2 \mathbf{W}_{0,0} \tag{30}$$

The amplitude of the anti-rotating excitation, a medium atom excited with two photons present can be shown to be

$$b_{mkk}(k) = \lim_{t\to\infty} \sqrt{NV}\mu^2 |g_k|^2 \frac{\exp\left[-i\left(\hbar\omega_k^r + \left(1-\frac{1}{2}N\alpha\right)\omega_k\right)t\right]}{\omega_k^r + \left(1-\frac{1}{2}N\alpha\right)\omega_k - \omega_0 + in\gamma} \times \frac{1}{\left(1-\frac{1}{2}N\alpha\right)\omega_k - \omega - 2\omega_k}\mathbf{W}_{m,0} \tag{31}$$

and the corresponding probability, summing over different magnetic sublevels,

$$\sum_{m,k} |b_{mkk}(k)|^2 = NV\mu^4 \sum_k \frac{|g_k|^4}{\left[\omega_k/n - (\omega_0 - n^2\omega_0^r)\right]^2 + n^2\gamma^2} \times \left[\frac{1}{\left(1 - \frac{1}{2}N\alpha\right)\omega_k - \omega - 2\omega_k}\right]^2 \mathbf{W}_{0,0} \tag{32}$$

The probability $\sum_{m,k} |b_m(k)|^2$ and $\sum_{m,k} |b_{mkk}(k)|^2$ are of order $N\alpha\omega_k/\omega$, which are negligible with $\omega_k \ll \omega$ and $N\alpha \ll 1$. By now we have shown all the nonvanishing amplitudes of the system at $\tau \to \infty$. The total probability, sum of equations (26)(30)(32), is found to be one to the order of $N\alpha$.

The average recoil energy of source atom is calculated solely from the amplitude b_k (24) as

$$\left\langle \frac{\hbar^2 k^2}{2M} \right\rangle = \sum_k \frac{\hbar^2 k^2}{2M} |b_k|^2 = n^2 \hbar \omega_0^r \tag{33}$$

This result shows that the source atom recoils according to the canonical photon momentum nk_0. This modification of the source atom recoil is directly related to the fact that the photon in the medium centers at a frequency $n\omega_0 + O\left(\hbar\omega_0^2/Mc^2\right)$ instead of $\omega_0 + O\left(\hbar\omega_0^2/Mc^2\right)$, as is shown in formula (24). In the spontaneous decay of the source atom, momentum of the source atom plus the field is conserved. This modification of the center freqency of the photon therefore results in a modification of the source atom recoil. However, this shift in the photon frequency seems strange, because we expect the frequency to center around the atomic frequency ω_0 from a energy conservation consideration. In the following, we give a detailed analysis of the energy conservation to order of the shifted frequency, namely, $n\hbar\omega_0 - \hbar\omega_0 = \frac{1}{2}N\alpha\hbar\omega_0$.

Before we proceed to show energy conservation, it is necessary to note that the source atom excited at $t = 0$ is not an eigenstate of the system. However, a discussion of energy conservation is meaningful in the context of average energy being conserved.

In order to find all the other forms of energy besides the photonic excitation, we should note that as the photon moves along in the medium, medium atoms inside the sphere of $R = ct$ are excited, according to $e^{-\gamma(ct-R)}$. Even the excitation probability is of order of magnitude $N\alpha\omega_k/\omega$, as far as the energy is concerned, this produces a correction of order $N\alpha\omega_k$. Including this contribution to the energy, and the interaction energy of the medium atoms with the field, we should be able to recover the conservation of energy. In the following

we work out explicitly energies associated with different excitations in the system and show that the total average energy is conserved.

The first part of the energy corresponding to the field excitation is

$$\sum_k |b_k|^2 \hbar\omega_k = \left(1 + \frac{1}{2}N\alpha\right)\hbar\omega_0 \tag{34}$$

The one associated with the medium atoms excitations is

$$\begin{aligned}&\sum_{m,k} |b_{mk}|^2 \hbar\omega + \sum_{m,k} |b_{mkk}|^2 (\hbar\omega + 2\hbar\omega_k)\\ &= -NV\sum_k \hbar^4 |g_k|^4 \mathbf{W}_{00}\frac{1}{(\hbar\omega_k/n - \hbar\omega_0)^2 + \gamma^2}\frac{1}{\Delta'}\end{aligned} \tag{35}$$

where

$$\frac{1}{\Delta'} \equiv -\hbar\omega_m\left\{\left[\frac{1}{\left(1-\frac{1}{2}N\alpha\right)\hbar\omega_k - \hbar\omega}\right]^2 + \left[\frac{1}{\left(1+\frac{1}{2}N\alpha\right)\hbar\omega_k + \hbar\omega}\right]^2\right\}$$

Here the difference in the energies of state b_{mk} and b_{mkk} can be ignored, because it amounts to a correction in the order $N\alpha\omega_k/\omega$. To leading order of ω_k/ω_m, we find that

$$\frac{1}{\Delta'} - \frac{1}{\Delta(E)} = -\frac{1}{\hbar\omega}N\alpha\frac{\omega_k}{\omega} \tag{36}$$

which enables us to neglect the difference between $\frac{1}{\Delta'}$ and $\frac{1}{\Delta(E)}$. Here the detuning $\Delta(E)$ is evaluated at the pole of the photon propagator $E = \left(1-\frac{1}{2}N\alpha\right)\hbar\omega_k$. Making use of the definition (21), one finds that the energy associated with the atomic excitations is $\frac{1}{2}N\alpha\hbar\omega_0$. The third part of the energy, the interaction energy between excited medium atoms and the field can be calculated as,

$$\begin{aligned}\langle V\rangle &= \sum_{m,k} NV\hbar g_k\left(b_k^* b_m + b_k^* b_{mkk}\right) + h.c.\\ &= \sum_k -N\alpha(\omega_k)\hbar\omega_k\frac{|\hbar g_k|^2}{\left(\hbar\omega_k/n - \hbar\omega_0 + n^2\hbar\omega_k^r\right)^2 + n^2\hbar^2\gamma^2}\\ &= -N\alpha(\omega_k)\hbar\omega_k\end{aligned} \tag{37}$$

adding up all of these contributions, we have

$$\sum_k |b_k|^2 \hbar\omega_k + \sum_{m,k}\left(|b_{mk}|^2 + |b_{mkk}|^2\right)\hbar\omega + \langle V\rangle = \hbar\omega_0 \tag{38}$$

showing that the frequency shift of the photon in the medium is compensated by the excitation energy of medium atoms and the interaction energy of excited atoms with the field.

III. DISCUSSION

In this paper we have shown that the source atom recoils according to $n\hbar k_0$, which agrees with the previous theoretical and experimental results [2][3][6]. This modification of the photon recoil arises in our calculation as a result of the interference of the different scattering amplitudes of the photon. As has been shown in the calculation, while the center frequency of the soure atom is at ω_0, only frequencies centered at $n\omega_0$ experience contructive interference. This is very much similar to the case of a source atom radiating in a cavity with the cavity frequency instead of the atomic frequency. From a quantum point of view, the source atom decays because it radiates and reabsorbs virtual photons. Such a process introduces a finite self energy whose real part gives the level shift and whose imaginary part gives the decay of the atomic excitation [7][8][12]. This process, though not included explicitly in our calculation, is the only way that the source atom "knows about" the enviroment(vacuum, cavity or dielectric medium) in which it locates. In the cavity, the virtual photon can interfere with those reflected from the cavity walls and constructive interference occurs only at the cavity freqency. In the dielectric medium the virtual photons radiated by the source atom can be scattered by medium atoms and reabsorbed by the source atom during the time $t \preceq 1/\gamma$. Different scattering amplitudes interfere to shift the real radiating frequency to $n\omega_0$. This is a different effect from the level shift, the real part of the source atom self energy, due to the interaction of the source atom with the enviroment. In our particular example of the source atom located in a dilute medium, the shift due to interacting is of the order $N\alpha\gamma$ while the shift due to interference is of the order $N\alpha\omega_0$.

An alternative explaination can be put forward in terms of eigen excitations of the system. Actually, if we consider the interaction of the field with the medium atom, the eigenmode of the system is neither medium atoms being excited nor a photon present, but a superposition of these two type of excitations, or a polariton[15]. The source atom decays by radiating polaritons instead of photons. For the energy to be conserved, the polariton energy plus the recoil energy should equal to the initial average energy $\hbar\omega_0$. As we have shown, the photon carries only part of the energy of the eigen excitation. On the other hand, the momentum carried by the medium atoms is negligible, the polariton momentum is just the photon momentum. When we require energy and momentum conservation for

the radiating process, the only possibility is the source atom to recoil according to $n\hbar k_0$.

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[10] To find the resonant frequency from b_k, we need to solve the equation $n\frac{\hbar\omega_k^2}{2mc^2} + \omega_k - n\omega_0 = 0$. This equation has two solutions, one at $\omega_k = n\left(\omega_0 - \frac{\hbar(nk_0)^2}{2m}\right)$, and the other at $\omega_k = -\frac{2mc^2}{n\hbar}$. The second is irrelevant and we simply ignore it.

[11] We have not included the local field corrections to the index of refraction. These corrections correspond to higher order corrections in terms of $N\alpha$.

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[16] The integral (26) can also be done by completing the contour at the lower half plane. The pole we pick up in this case is $n\left(\omega_0 - n^2\omega_0^r - in\gamma\right)$. This procedure yields the same result.